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A note on head-sea diffraction by a slender body

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In studying head-sea diffraction by a slender body, Haren & Mei (1981) found, at a special frequency, a singularity inherent in the cross-section (inner) mathematical problem. In this note this singularity is interpreted, and it is shown that the head-sea problem is similar to the problem of an acoustic duct being excited at one end by a pressure field. As a consequence, the head sea is not always refracted away from the body, since the solution remains oscillatory for frequencies above that special frequency.

1. Introduction

Several different researchers, using different methods, have established that a head sea is refracted away by a long ship, leaving a comparatively wave-free zone near the body (see e.g. Faltinsen 1971; Ursell 1977; Haren & Mei 1981). In the latter paper it was found numerically that the cross-section (inner) problem is singular at some special frequency. Later Yue & Mei (1981) showed that this singularity is inherent in the inner problem and that it is closely related to the non-existence at this frequency of the solution derived by Ursell (1968). Owing to this fact, this special frequency will here be called 'Ursell's frequency'.

In this note this singularity is interpreted, and it is shown that at Ursell's frequency the homogeneous cross-section problem has a non-trivial solution. The situation here is similar to the acoustic-duct problem and its cut-off frequencies. At these special values, the homogeneous cross-section problem has a non-trivial solution, and, as a consequence, the wave propagated along the duct has a definitive change of behaviour. In short, for frequencies below the cut-off frequency the wave mode goes to zero as $x \to \infty$, while for frequencies above cut-off the wave mode remains oscillatory. If a velocity field is prescribed at the end of the acoustic duct, the transition is abrupt (resonance). If, instead, a pressure field is given, the transition is smooth. The same sort of behaviour is expected at Ursell's frequency. As a matter of fact, it is shown here that the head-sea problem is similar to that of the acoustic duct when pressure is imposed at one end.

In what follows, the cross-section problem will be analysed first for the shallow-water limit and then for the arbitrary-water-depth case. There is an important reason for doing this. The solution of the shallow-water equation can be determined analytically, and the features of the cross-section problem can be disclosed very easily. It can be shown that the same sort of results can also be derived for the arbitrary-water-depth case, but then the demonstration is much more complex. In this paper the shallow-water equation is invoked to motivate these results, and only a sketchy view of the arbitrary-water-depth case, with discussion of pertinent results, will be given. Details can be found in Aranha (1982).

Once the cross-section (inner) solution is obtained and analysed, the parabolic approximation will be used as an outer solution. The matching between the two produces an Abel integral equation, which, for a constant cross-section, has an explicit solution, as given by Mei & Tuck (1980). This solution will be analysed, and the change of behaviour at Ursell's frequency will be demonstrated.

2. Cross-section (inner) problem

The geometric parameters are defined in figure 1. In this work specific reference will be made to the rectangular box indicated, although the conclusions are general. In figure 1, $\bar{b}$ exceeds b, but is otherwise arbitrary. The equations for shallow water depend on b, and $\bar{b}$ will take its place for arbitrary water depth.

The incident wave is given by $\phi_{\mathrm{I}}(x, y, z) = A_{\mathrm{I}}\,\mathrm{e}^{\mathrm{i}K_0 x}\cosh K_0(z+h)$, where A_{I} is the wave amplitude and K_0 the wavenumber. The total wave is defined as

$$\phi(x, y, z) = \phi_{\mathrm{I}}(x, y, z) + \psi(x, y, z)\,\mathrm{e}^{\mathrm{i}K_0 x}. \tag{1}$$

In the cross-section (inner) problem the diffracted potential $\psi(x, y, z)$ is independent of x. It follows then that it satisfies the set of equations

$$\psi_{yy} + \psi_{zz} - K_0^2\psi = 0 \quad \text{for } -h < z < 0, \tag{2a}$$

$$\psi_z = \frac{\omega_0^2}{g}\psi \quad \text{on } z = 0, \quad \text{where } \frac{\omega_0^2}{g} = K_0 \tanh K_0 h, \tag{2b}$$

$$\psi_z = 0 \quad \text{on } z = -h, \tag{2c}$$

$$\frac{\partial \psi}{\partial n} = -\frac{\partial}{\partial n}(\cosh K_0(z+h)) \quad \text{on body } B, \tag{2d}$$

$$\psi \rightarrow [A_0^{\pm} + K_0 U_0^{\pm}(|y| - \bar{b})]\cosh K_0(z+h) \quad \text{when } y \rightarrow \pm\infty. \tag{2e}$$

Condition (2e) is a little different from the one used by Ursell (1968). It is the most general one, and so is appropriate for matching with the outer solution. Equations (2a–e) are valid for arbitrary water depth, and will be particularized next for the shallow-water case.

2.1. *Shallow water*

The purpose here is to derive, in an easy way, the results that are valid for arbitrary depth. In this subsection the cross-section will be assumed to be rectangular, as indicated in figure 1, and the water to be shallow ($K_0 h \ll 1$). It follows that

$$\omega_0^2/g = K_0^2 h[1 + O(K_0 h)^2], \quad \phi_{\mathrm{I}}(x, y, z) = A_{\mathrm{I}}\,\mathrm{e}^{\mathrm{i}K_0 x}[1 + O(K_0 h)^2]$$

and, if $\psi_0(y) = \int_{-h}^{0} \psi(y, z)\,\mathrm{d}z$ then, with an error factor of the form $1 + O(K_0 h)^2$, (2) become (see Yue & Mei 1981)

$$\psi_0'' - K_0^2\psi_0 = A_{\mathrm{I}} K_0^2 \quad \text{for } |y| < b, \tag{3a}$$

$$(\psi_0^{\pm})'' = 0 \quad \text{for } y \gtrless \pm b, \tag{3b}$$

$$\psi_0^{\pm} \rightarrow A_0^{\pm} + K_0 U_0^{\pm}(|y| - b) \quad \text{as } y \rightarrow \pm\infty, \tag{3c}$$

where $\psi'' = \psi_{yy}$. Equation (3a) plays the role of the boundary condition on the body surface, and the total wave $\phi(x, y) = \mathrm{e}^{\mathrm{i}K_0 x}[A_{\mathrm{I}} + \psi_0(y)]$ satisfies the homogeneous boundary condition $\phi_{yy} - K_0^2\phi = 0$ ($|y| < b$).

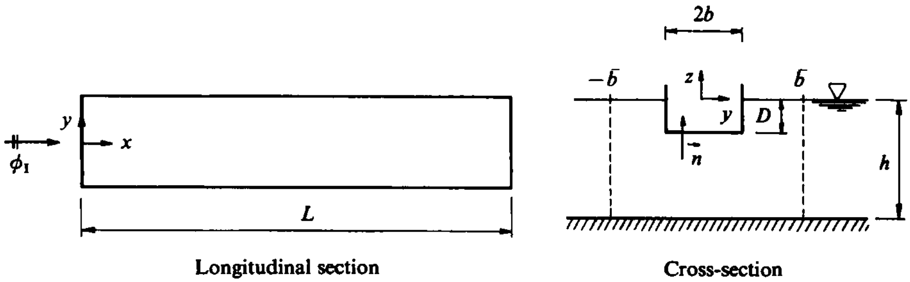

FIGURE 1. Notation.

The solution of (3) must satisfy the continuity of pressure and flux at $|y| = b$ $(\psi_0(\pm b) = \psi_0^\pm(\pm b);\ (h-D)\,\psi_{0,y}(\pm b) = h\psi_{0,y}^\pm(\pm b))$ and is given by

$$\psi_0(y) = -A_I + C\cosh K_0 y + S\sinh K_0 y \quad (|y| < b), \tag{4a}$$

$$\psi_0(y) = [A_0^\pm + K_0 U_0^\pm(|y|-b)] \quad (|y| > b), \tag{4b}$$

where $A_0^\pm$ and $U_0^\pm$ are related by means of

$$\begin{bmatrix} a_1 & a_2 \\ a_2 & a_1 \end{bmatrix}\begin{bmatrix} A_0^+ \\ A_0^- \end{bmatrix} = A_I\begin{bmatrix} V_0^+ \\ V_0^- \end{bmatrix} + \begin{bmatrix} U_0^+ \\ U_0^- \end{bmatrix} \tag{5}$$

and

$$\left.\begin{aligned} a_1 &= \frac{1}{2}\frac{h-D}{h}(\coth K_0 b + \tanh K_0 b), \\ a_2 &= \frac{1}{2}\frac{h-D}{h}(-\coth K_0 b + \tanh K_0 b), \\ V_0^\pm &= -(a_1+a_2) = -\frac{h-D}{h}\tanh K_0 b. \end{aligned}\right\} \tag{6}$$

The matrix in (5) is symmetric, positive-definite and it has two positive eigenvalues,

$$\left.\begin{aligned} \lambda_1 &= a_1 + a_2 = \frac{h-D}{h}\tanh K_0 b, \\ \lambda_2 &= a_1 - a_2 = \frac{h-D}{h}\coth K_0 b. \end{aligned}\right\} \tag{7}$$

If the cross-section is symmetric (as it is in this case) the physical solution must also be. Taking $S = 0$, $A_0 = A_0^\pm$, $U_0 = U_0^\pm$ in (4) and (5), the following result is obtained:

$$\psi_0(y) = -A_I + C\cosh K_0 y \quad (|y| < b), \tag{8a}$$

$$\psi_0(y) = (A_0 - (K_0 b)\,U_0) + K_0 U_0|y| \quad (|y| > b), \tag{8b}$$

$$A_0 + A_I = \frac{U_0}{\lambda_1}. \tag{8c}$$

The total potential can be written as

$$\phi(x,y) = C\cosh K_0 y\, e^{iK_0 x} \quad (|y| < b), \tag{9a}$$

$$\phi(x,y) = \bar{A}_I e^{iK_0 x} + K_0 U_0 |y|\, e^{iK_0 x} \quad (|y| > b), \tag{9b}$$

where
$$\bar{A}_{\mathrm{I}} = A_{\mathrm{I}} + A_0 - (K_0 b)\, U_0 = \frac{U_0}{\lambda_1}(1-\bar{\lambda}_1), \tag{10a}$$

$$\bar{\lambda}_1 = (K_0 b)\,\lambda_1 = \frac{h-D}{h} K_0 b \tanh K_0 b, \tag{10b}$$

$$C = \frac{1}{\lambda_1} \frac{U_0}{\cosh K_0 b}. \tag{10c}$$

Notice that the inner solution (9) is always bounded ($\lambda_1 > 0$), and it depends on U_0. This value will be determined by matching with the outer solution. It is easy to see also that $\phi(x,y) = 0$ if $U_0 = 0$.

Ursell (1968) specifies that $\psi_0(y) \to K_0 U_0 |y|$ when $|y| \to \infty$. From (8b) it follows that $A_0 = (K_0 b)\, U_0$, and from (8c) (see (10b)) that

$$(1-\bar{\lambda}_1)\, U_0 = \lambda_1 A_{\mathrm{I}}. \tag{11}$$

Ursell's solution does not exist for a $K_0 = K^*$ such that

$$\bar{\lambda}_1(K^*) = 1, \tag{12a}$$

or, in this case (see (10b)),

$$K^* b \tanh K^* b = \frac{h}{h-D}. \tag{12b}$$

Equation (12b) has one and only one solution $K^* b$ for each choice of $h/(h-D)$. The root K^* of (12a) is here called 'Ursell's frequency'.

A neat way to interpret the meaning of such K^* is provided by (9b): $\phi(x,y)$ is written there as the sum of an incident wave, with amplitude $\bar{A}_{\mathrm{I}}$, and the diffracted wave $K_0 U_0 |y| \mathrm{e}^{\mathrm{i}K_0 x}$. Since $\phi(x,y)$ satisfies the homogeneous boundary condition at the body surface, the diffracted potential is excited by the incident wave. The problem is said to be homogeneous when the excitation is zero ($\bar{A}_{\mathrm{I}} = 0$). From (10a) and (12a) it follows that at Ursell's frequency the homogeneous cross-section problem ($\bar{A}_{\mathrm{I}} = 0$) has a non-trivial solution (U_0 arbitrary).

The situation here is similar to the acoustic-wave-guide problem, Ursell's frequency playing the role of the cut-off frequencies. But if this is so it should be suspected that a change of the asymptotic behaviour of ϕ, when $x \to \infty$, takes place. This will be demonstrated later in this work. Next the cross-section problem for the case where the water depth is arbitrary will be briefly addressed.

2.2. *Arbitrary water depth*

If the water is not shallow the solution must be obtained numerically. One convenient numerical algorithm is provided by the hybrid-element method, first introduced by Chen & Mei (1974). The basic idea is the following. In the uniform region $|y| > \bar{b}$ the solution is expressed as a series with unknown coefficients. In the region $|y| < \bar{b}$, where the scatterer is placed, the differential operator of (2) is transformed to its weak form. Imposing continuity of pressure and velocity along $|y| = \bar{b}$, a weak equation is obtained, defined only in the finite region $|y| < \bar{b}$. The exact solution of this weak equation is then approximated, via finite elements in a standard way.

The series expansion associated with (2a–c), is given by

$$\psi^{\pm}(y,z) = [A_0^{\pm} + K_0 U_0^{\pm}(|y|-\bar{b})] \cosh K_0(z+h) + \sum_{n=1}^{\infty} A_n^{\pm} f_n(z)\, \mathrm{e}^{-\beta_n(|y|-\bar{b})}, \tag{13}$$

where
$$\left.\begin{aligned}\beta_n &= (K_n^2+K_0^2)^{\frac{1}{2}},\\ f_n(z) &= \cos K_n(z+h), \quad \frac{\omega_0^2}{g} = -K_n \tan K_n h.\end{aligned}\right\} \tag{14}$$

Expression (13) is valid for an arbitrarily large (although always finite) water depth.

The important point to be emphasized is that the arbitrary-water-depth case can be cast in a form identical with the shallow-water limit. In particular, it can be shown (see Aranha (1982)) that

(i) $U_0^{\pm}$ and $A_0^{\pm}$ are related by an expression like (5), where the matrix is again positive-definite and $V_0^{\pm} = -\lambda_1 = -(a_1+a_2)$;

(ii) $\phi(x,y,z) = 0$ if $U_0^{\pm} = 0$, or, in other words, $\phi(x,y,z)\rightarrow 0$ if $K_0\, U_0^{\pm} \rightarrow 0$,

(iii) as in (8*c*) and (10*a*, *b*),
$$\left.\begin{aligned}A_0+A_{\mathrm{I}} &= \frac{U_0}{\lambda_1},\\ \bar{A}_{\mathrm{I}} = A_{\mathrm{I}}+A_0-(K_0\bar{b})\,U_0 &= \frac{U_0}{\lambda_1}(1-\bar{\lambda}_1),\\ \bar{\lambda}_1 &= (K_0\bar{b})\,\lambda_1,\end{aligned}\right\} \tag{15}$$
where $A_0 = A_0^{\pm}$ and $U_0 = U_0^{\pm}$, once it is assumed that the cross-section is symmetric;

(iv) Ursell's solution does not exist for a $K_0 = K^*$ such that
$$\bar{\lambda}_1(K^*) = 1. \tag{16}$$
At these Ursell frequencies, the homogeneous cross-section problem ($\bar{A}_{\mathrm{I}} = 0$) has a non-trivial solution.

The only difference between the general case and the shallow-water limit is that now the exact values of a_1, a_2 and $V_0^{\pm}$ are unknown and must be computed numerically. The consequence of this is that here the existence and uniqueness of the solution for (16) is not granted, as it has been before (see (12*b*)). However, an upper bound for $\bar{\lambda}_1 = \bar{\lambda}_1(K_0)$ can be derived, and this helps to visualize the behaviour of Ursell's frequencies. In fact it can be shown that if the water is not shallow ($K_0 h > 0.17587$) then (see equations (8)–(22) of Aranha 1982)
$$\left.\begin{aligned}\bar{\lambda}_1(K_0) < f(K_0) &= 2(K_0 b)^2\, I(K_0;B),\\ I(K_0;B) &= \frac{1}{2b}\int_B \mathrm{e}^{-2K_0|z|}\, n_z\, \mathrm{d}z,\end{aligned}\right\} \tag{17}$$
where $2b$ is the beam, B is the contour-line of the cross-section and n_z the z-component of the normal $\boldsymbol{n}$ (see figure 1). This result has been deduced under the assumption that the projection of the cross-section in the free surface coincides with the waterline. In this case the integral $I(K_0;B)$, which depends only on the wavenumber and geometry of the cross-section, is always positive.

To get a clear picture, the upper bound above will be specialized for some common sections.

(*a*) *Rectangular box*

In this case
$$\bar{\lambda}_1(K_0) < 2(K_0 b)^2\, \mathrm{e}^{-2K_0 D},$$
and then (16) has no solution if $\gamma = D/2b > 0.2601$. Also the number of roots must be even when $D > 0$. If $(K_1^*, K_2^*, \ldots, K_{2n}^*)$ are these solutions then $K_1^* b > \frac{1}{2}\sqrt{2} \approx 0.707$, since $\bar{\lambda}(K_0) < 2(K_0 b)^2$.

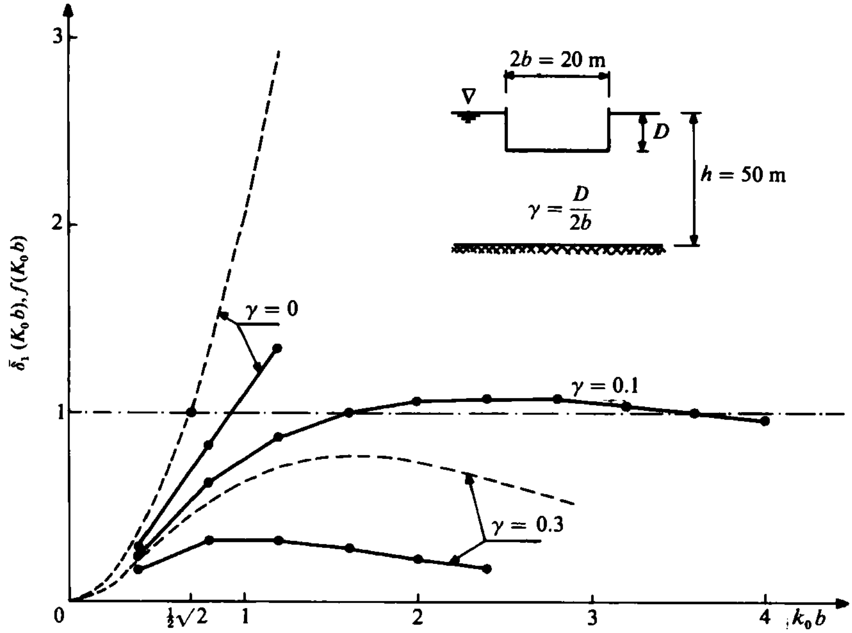

FIGURE 2. Ursell's frequency. ●——●, $\bar{\lambda}_1(K_0 b)$; ---, $f(K_0 b)$ = upper bound.

The separation between the roots increases with b/D, and numerical evidence indicates that there should exist only one root if $D = 0$ (see Yue & Mei 1981). Figure 2 displays some numerically computed values of $\bar{\lambda}_1(K_0)$, and they confirm the conclusions quoted above.

(*b*) *Half-immersed circle*

Now $I(K_0; B) = 1 - \frac{1}{2}\pi(I_1(2K_0 b) - L_1(2K_0 b))$, where $I_1(\ )$ and $L_1(\ )$ are the modified Bessel and Struve functions (see Abramowitz & Stegun 1964). It can easily be seen that $f(K_0 b) = 2(K_0 b)^2 [\frac{1}{2}\pi(I_1(2K_0 b) - L_1(2K_0 b))]$ is smaller than 1 for any $K_0 b$. Equation (16) has no roots in this case.

From these results the following conclusions can be reached.

(i) If Ursell's frequency exists then (in general) it is not unique.

(ii) Ursell's frequency should not exist if the body has a 'large' draught. Indeed, for a rectangular box this happens if $\gamma = D/2b > 0.2601$, and for a half-immersed circle ($\gamma = 0.5$) there is no Ursell frequency.

(iii) If the water is not shallow ($K_0 h > 0.17587$) all these results do not depend on the water depth, since they have been derived from inequality (17).

Suppose that $(K_1^*, K_2^*, \ldots, K_{2n}^*)$ are the Ursell frequencies and let

$$A = \bigcup_{j=1}^{n} \{K_{2j-1}^* < K_0 < K_{2j}^*\}.$$

To keep the notation short, a value K_0 will be said to be above (below) the Ursell frequency if $K_0 \in A$ (or $K_0 \notin A$).

In any event, whether Ursell's frequencies exist or not, it follows from (13), (15) and (1) that

$$\phi(x, y, z) \sim [\bar{A}_{\mathrm{I}} + K_0 U_0 |y|] \cosh K_0(z+h)\, e^{iK_0 x} \quad \text{as } |y| \to \infty. \tag{18}$$

Expression (18) will be used for matching with the outer solution. Notice here that it agrees with (9*b*) when the water is shallow, with the error factor $1 + O(K_0 h)^2$.

3. Parabolic (outer) approximation and cut-off frequency

If the wave is short ($K_0 b = O(1)$) and the body is slender ($L/b \gg 1$) the total wave can be approximated, far from the body, as a modulation of the head sea. That is,

$$\phi(x,y,z) \sim P(x,y)\, A_{\mathrm{I}}\, \mathrm{e}^{\mathrm{i}K_0 x} \cosh K_0(z+h). \tag{19}$$

The modulation $P(x,y)$ satisfies a parabolic equation similar to the heat equation. The appropriate solution must satisfy some initial and boundary conditions. Details can be found in Mei & Tuck (1980) and Haren & Mei (1981). In these papers it is shown that $P(x,y)$ is given by

$$P(x,y) = 1 - \frac{1+\mathrm{i}}{2(\pi K_0)^{\frac{1}{2}}} \int_0^x \frac{\mathrm{d}\xi\, V(\xi)}{(x-\xi)^{\frac{1}{2}}} \exp\left[\frac{(\mathrm{i}K_0 y^2)}{2(x-\xi)}\right], \tag{20}$$

where the unknown $V(x)$ must be determined by matching with the inner solution. Placing (20) into (19) and taking the limit as $|y| \rightarrow 0$, the following asymptotic approximation of the parabolic (outer) solution is obtained:

$$\phi(x,y,z) \sim [(1-I(V)) + V(x)\,|y|\,]\, A_{\mathrm{I}} \cosh K_0(z+h)\, \mathrm{e}^{\mathrm{i}K_0 x} \quad \text{as } |y| \rightarrow 0, \tag{21}$$

where
$$I(V) = \frac{1+\mathrm{i}}{2(\pi K_0)^{\frac{1}{2}}} \int_0^x \frac{\mathrm{d}\xi\, V(\xi)}{(x-\xi)^{\frac{1}{2}}}.$$

The outer expansion (18) of the inner solution must be matched with the inner expansion (21) of the outer solution. From this matching it follows that $V(x)$ must satisfy the Abel integral equation

$$1 - \frac{1+\mathrm{i}}{2(\pi K_0)^{\frac{1}{2}}} \int_0^x \frac{\mathrm{d}\xi\, V(\xi)}{(x-\xi)^{\frac{1}{2}}} = \sigma V(x), \tag{22}$$

with
$$\sigma = \frac{1}{K_0 \lambda_1}(1-\bar{\lambda}_1), \tag{23a}$$

$$K_0 U_0 = A_{\mathrm{I}}\, V(x). \tag{23b}$$

In the shallow-water limit the solution can be written as (see (7), (9*a*), (10*c*))

$$\phi(x,y) = A_{\mathrm{I}} \frac{h}{K_0(h-D)} V(x) \frac{\cosh K_0 y}{\sinh K_0 b} \mathrm{e}^{\mathrm{i}K_0 x} \quad (|y| < b). \tag{24}$$

Equation (24) shows that the behaviour of $\phi(x,y)$ as $x \rightarrow \infty$ depends solely on $V(x)$. The same is true for arbitrary water depth, since $\phi(x,y,z) \rightarrow 0$ if $K_0 U_0 = A_{\mathrm{I}} V(x) \rightarrow 0$.

When the cross-section is constant, σ is also, and the solution $V(x)$ can be obtained explicitly. In fact (see Mei & Tuck 1980)

$$V(x) = -\frac{1}{|\sigma|} \exp\left(\frac{\mathrm{i}x}{2K_0|\sigma|^2}\right) \left\{\operatorname{erf}\left(\frac{(1+\mathrm{i})\, x^{\frac{1}{2}}}{2K_0^{\frac{1}{2}}|\sigma|}\right) - \frac{\sigma}{|\sigma|}\right\}, \tag{25}$$

where $\operatorname{erf}(z)$ is the error function. The asymptotic behaviour of (25) is given by (see Abramowitz & Stegun 1964)

$$V(x) \sim -\frac{1}{|\sigma|}\left(1 - \frac{\sigma}{|\sigma|}\right) \exp\left(\frac{\mathrm{i}x}{2K_0\sigma^2}\right) + \frac{1-\mathrm{i}}{\pi^{\frac{1}{2}}} \frac{K_0}{(K_0 x)^{\frac{1}{2}}}\left[1 + O\left(\frac{K_0\sigma^2}{x}\right)\right] \tag{26}$$

when $x/K_0\sigma^2 \rightarrow \infty$. Equation (26) shows that the behaviour of $V(x)$ as $x \rightarrow \infty$ depends on the sign of σ. From (23*a*) and (16) σ changes sign at Ursell's frequency. If $\sigma > 0$

(frequency below Ursell's frequency) $V(x)$ (and so $\phi(x,y,z)$ also) tends to zero with $(K_0 x)^{\frac{1}{2}}$ as $x\to\infty$. This is the result obtained by Faltinsen (1971), Ursell (1977) and Haren & Mei (1981). If $\sigma<0$ (frequency above the Ursell frequency) the first term in (26) dominates for large x. It is clear then that $\phi(x,y,z)$ remains oscillatory as $x\to\infty$. So Ursell's frequency works as if it were the classical cut-off frequency in a waveguide.

It remains to understand what happens at $\sigma=0$. From (26) it is clear that $V(x)$ is bounded when $\sigma\to 0$ from above, with the exception of the integrable singularity at $x=0$. When $\sigma\to 0$ from below, there is an extra term,

$$G(x,\sigma)=-\left(\frac{2}{|\sigma|}\right)\exp\left(\frac{ix}{2K_0\sigma^2}\right).$$

As $|\sigma|\to 0$, $|G(x,\sigma)|\to\infty$, but the phase is rapidly oscillatory. There is no doubt that the solution of the integral equation (22) is unbounded as $\sigma\to 0-$, but the same conclusion cannot be made about $\phi(x,y,z)$. In fact the rapidly oscillatory nature of the phase makes necessary to interpret this limit in the generalized sense. Equation (24), for shallow water, provides an easy way to show this. If $K_0\Delta x\ll 1$ let $\bar{\phi}(x,y;-|\sigma|;\Delta x)$ be the average value of $\phi(x,y;-|\sigma|)$ in the interval $x-\Delta x<\xi<x+\Delta x$. For $x>0$ and using (26), the following equality is obtained for $|\sigma|\ll 1$:

$$\begin{aligned}\bar{\phi}(x,y;-|\sigma|;\Delta x)&\\ &=\frac{1}{2\Delta x}\int_{x-\Delta x}^{x+\Delta x}\phi(\xi,y;-|\sigma|)\,d\xi\\ &\sim A_{\mathrm{I}}\frac{h}{K_0(h-D)}\frac{\cosh K_0 y}{\sinh K_0 b}\,e^{iK_0 x}\left\{-4K_0|\sigma|\exp\left(i\frac{x}{2K_0\sigma^2}\right)\left[\frac{1}{\Delta x}\sin\left(\frac{\Delta x}{2K_0\sigma^2}\right)\right]\right.\\ &\left.+\frac{1-i}{\pi^{\frac{1}{2}}}\frac{K_0}{(K_0x)^{\frac{1}{2}}}\left[1+O\left(\frac{K_0\sigma^2}{x}\right)\right]\right\}\left[1+O(K_0\Delta x)\right].\end{aligned}$$

Therefore

$$\bar{\phi}(x,y,0-;\Delta x)\sim A_{\mathrm{I}}\frac{h}{K_0(h-D)}\frac{\cosh K_0 y}{\sinh K_0 b}\frac{1-i}{\pi^{\frac{1}{2}}}\frac{K_0}{(K_0x)^{\frac{1}{2}}}e^{iK_0x}[1+O(K_0\Delta x)].$$

Notice that $\bar{\phi}(x,y,0-,\Delta x)=\bar{\phi}(x,y,0+;\Delta x)$. The limit, as $|\sigma|\to 0$, in the sense of generalized functions, assumes that $\phi(x,y;\sigma=0)=\lim_{\Delta x\to 0}\bar{\phi}(x,y;0,\Delta x)$, or

$$\phi(x,y;\sigma=0)=A_{\mathrm{I}}\frac{h}{h-D}\frac{\cosh K_0 y}{\sinh K_0 b}\frac{1-i}{\pi^{\frac{1}{2}}}\frac{1}{(K_0x)^{\frac{1}{2}}}e^{iK_0x}. \tag{27}$$

This limit can be physically interpreted in the following way. Since the pressure $p(x,y;-|\sigma|)$, is proportional to the potential, let $p(x,y;-|\sigma|)=\phi(x,y;-|\sigma|)$. The force acting in a small neighbourhood around the point (x,y) is given by

$$F(x,y;-|\sigma|;\Delta x)=\bar{\phi}(x,y;-|\sigma|;\Delta x)(\Delta x)^2[1+O(K_0\Delta x)].$$

As $|\sigma|\to 0$,

$$F(x,y;-|\sigma|;\Delta x)\to F(x,y;0;\Delta x)=\bar{\phi}(x,y;0;\Delta x)(\Delta x)^2[1+O(K_0\Delta x)].$$

By definition, pressure is the limit of force over area. So

$$p(x,y;\sigma=0)=\lim_{\Delta x\to 0}(\Delta x)^{-2}F(x,y;0;\Delta x),$$

and (27) holds, since $p(x,y;\sigma=0)=\phi(x,y;\sigma=0)$.

It turns out that the transition at the cut-off frequency (Ursell's frequency) is smooth. The head-sea problem is similar to the acoustic-duct problem when the pressure field is imposed as a boundary condition, since there is no resonance at this frequency.

4. Conclusion

The singularity at Ursell's frequency has been interpreted. It has been shown, as in the classical waveguide problem, that the existence of characteristic frequencies for which the homogeneous cross-section problem has a non-trivial solution, makes its presence felt in the asymptotic behaviour of the wave as $x \rightarrow \infty$. For shallow water there exists only one Ursell frequency, but for finite depth there exist an even number of such frequencies if the draught is non-zero. For frequencies below the first Ursell frequency the wave decays with x, but, if the frequency is above the first and below the second, the wave remains oscillatory as $x \rightarrow \infty$. Each time an Ursell frequency is crossed, a change of behaviour like this must happen. The results obtained for the rectangular box indicate that if the draught is large Ursell's frequency should not exist, as it does not for a half-immersed circle. In other words, a head sea is in this case always refracted away by a long ship, a result in accordance with Ursell (1977).

The interpretation of the limit $(2/\sigma)\exp(\mathrm{i}x/2K_0\sigma^2)$ in the generalized sense, and the physical meaning of such a limit, poses a question about the resonance found by Mei & Tuck (1980) in another problem.

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